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Quantum Gravity on dS_3

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Abstract

We study quantum gravity on dS_3 using the Chern-Simons formulation of three-dimensional gravity. We derive an exact expression for the partition function for quantum gravity on dS_3 in a Euclidean path integral approach. We show that the topology of the space relevant for studying de Sitter entropy is a solid torus. The quantum fluctuations of de Sitter space are sectors of configurations of point masses taking a *discrete* set of values. The partition function gives the correct semi-classical entropy. The sub-leading correction to the entropy is logarithmic in horizon area, with a coefficient -1. We discuss this correction in detail, and show that the sub-leading correction to the entropy from the dS/CFT correspondence agrees with our result. A comparison with the corresponding results for the AdS_3 BTZ blackhole is also presented.

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1 Introduction

It is well-known that for a timelike observer in deSitter space, there exists a horizon, and regions of spacetime beyond it are not accessible to this observer. The thermodynamics of such a cosmological horizon is very similar to that of black hole horizons [1]. In particular, there is an entropy associated with the de Sitter spacetime, which is a measure of the information loss (for the timelike observer) across the cosmological horizon. There have been several attempts to describe this entropy in terms of microscopic degrees of freedom at the cosmological horizon [2, 3], in a Chern-Simons formulation [4] and more recently, by a CFT at past or future infinity [5, 6] motivated by the dS/CFT correspondence [7]. It is shown in [8] that the asymptotic symmetry group of quantum gravity in dS_3 is the Euclidean conformal group in two dimensions. This has led to the proposal of a correspondence between quantum gravity in dS_3 and a CFT at asymptotic infinity. In [5], the entropy of dS_3 is computed by applying Cardy's formula for the growth of states in the asymptotic CFT - but as discussed there, there are many subtleties involved. Further, it is not clear how the entropy associated with information loss across the horizon is described by states at asymptotic infinity.

We study quantum gravity on dS_3 using the Chern-Simons formulation of Euclidean gravity in three dimensions which is described in section 3. Euclidean dS_3 has the topology of a three-sphere - we show, however, that the topology of the space that is relevant for studying the degrees of freedom that contribute to entropy is that of a solid torus. The degrees of freedom could be thought of as living on the boundary torus. The existence of degrees of freedom on the boundary has been studied earlier in the context of black holes in several approaches, for example [9, 10, 11, 12, 13]. Three dimensional Euclidean gravity with a positive cosmological constant can be described by two $SU(2)$ Chern-Simons theories [14, 15]. Then, $SU(2)$ Wess-Zumino conformal field theories are naturally induced on the boundary [16]. The quantum degrees of freedom corresponding to the de Sitter entropy are described by these conformal field theories. In section 4, we derive an *exact* expression for the canonical partition function for dS_3 in a Euclidean path integral approach. Considerations of gauge invariance necessitate a *discrete* sum in the partition function over point mass configurations as well, upto a certain maximum value of the mass. The significance of this discrete sum in the partition function will be discussed in Section 5. The partition function gives the correct semi-classical entropy for de Sitter space. The next-order correction to the entropy is logarithmic in the horizon "area" [17]. In the last section, we comment on the logarithmic correction to the semi-classical entropy. We compare the coefficient of this correction with that obtained using the dS/CFT correspondence and find that they agree. We also make a detailed comparison with similar logarithmic terms in the entropy of the BTZ black hole. The comparison suggests a connection between the regime considered in the black hole parameter space, and the coefficient of the logarithmic correction.

2 de Sitter gravity as a Chern-Simons theory

The gravity action I_{grav} in three dimensions written in a first-order formalism (using triads e and spin connection ω) is the difference of two Chern-Simons actions. For Lorentzian gravity with a positive cosmological constant,

$$I_{\rm grav} \;=\; I_{\rm CS}[A] \;-\; I_{\rm CS}[\bar{A}], \tag{1}$$

where

$$A \;=\; \left(\omega^a + \frac{i}{l}\, e^a\right) T_a, \qquad \bar{A} \;=\; \left(\omega^a - \frac{i}{l}\, e^a\right) T_a \tag{2}$$

are $\mathrm{SL}(2,\mathbf{C})$ gauge fields (with $T_a = -i\sigma_a/2$). Here, the positive cosmological constant $\Lambda = (1/l^2)$. The Chern-Simons action $I_{\rm CS}[A]$ is

$$I_{\rm CS} \;=\; \frac{k}{4\pi}\int_M \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) \tag{3}$$

and the Chern-Simons coupling constant is $k = -l/4G$.

Now, for a manifold with boundary, the Chern-Simons field theory is described by a Wess-Zumino conformal field theory on the boundary. Under the decomposition

$$A \;=\; g^{-1}dg \;+\; g^{-1}\tilde{A}g \;, \tag{4}$$

the Chern-Simons action (3) becomes [18], [19]

$$I_{\rm CS}[A] \;=\; I_{\rm CS}[\tilde{A}] \;+\; kI^{+}_{\rm WZW}[g,\tilde{A}_z] \;, \tag{5}$$

where $I^{+}_{\rm WZW}[g,\tilde{A}_z]$ is the action of a chiral $SU(2)$ Wess-Zumino model on the boundary ∂M,

$$\begin{aligned} I^{+}_{\rm WZW}[g,\tilde{A}_z] \;&=\; \frac{1}{4\pi}\int_{\partial M}\mathrm{Tr}\left(g^{-1}\partial_z g\, g^{-1}\partial_{\bar{z}} g \;-\; 2g^{-1}\partial_{\bar{z}} g\tilde{A}_z\right) \\ &+\; \frac{1}{12\pi}\int_M \mathrm{Tr}\left(g^{-1}dg\right)^3 . \end{aligned} \tag{6}$$

The 'pure gauge' degrees of freedom g are now true dynamical degrees of freedom at the boundary.

We are interested in quantum gravity on dS_3. Global $(2+1)-d$ deSitter spacetime is described by the metric

$$ds^2 \;=\; -\,l^2 d\tau^2 + l^2\cosh^2\tau d\Omega^2 \tag{7}$$

Equal time sections of this metric are two-spheres, and there are no globally timelike Killing vectors. However, there does exist a timelike Killing vector in certain patches of this spacetime. Figure 1 shows the Penrose diagram of global deSitter space with these patches - II and

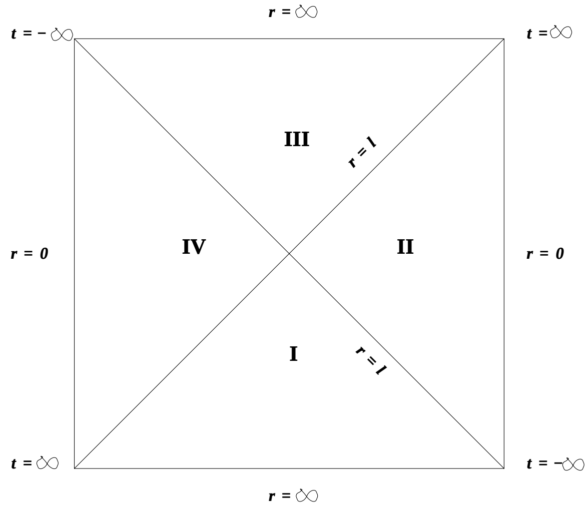

Figure 1.

IV. These regions are causally disconnected and the timelike Killing vector flows in opposite directions in these two patches. Each of these patches is bounded by the cosmological horizon, and descibed by the metric

$$ds^2 \;=\; -\,N^2\; dt^2 + N^{-2}\; dr^2 + r^2\; d\phi^2 \tag{8}$$

where

$$N^2 \;=\; (1 \;-\; \frac{r^2}{l^2}), \tag{9}$$

and $0 \le r \le l$. ϕ is an angular coordinate with period 2π. Since this metric is static, the patches II and IV are referred to as static patches. The cosmological horizon in these coordinates is therefore at $r = l$. Constant t surfaces are discs D_2, and the topology of the patch is $D_2 \otimes R$.

Using (2), the connection A^a corresponding to the metric (8) may be written as:

$$\begin{aligned} A^0 &= N(-d\phi + \frac{i}{l}dt) \\ A^1 &= \frac{i}{lN}dr \\ A^2 &= \frac{r}{l^2}dt + \frac{ir}{l}d\phi \end{aligned} \tag{10}$$

3 Euclidean de Sitter space

The Euclidean gravity action is the difference of two $SU(2)$ Chern-Simons actions - where the connections corresponding to the two actions are real and given by $A = \left(\omega^a + \frac{1}{l}\, e^a\right) T_a$ and $B = \left(\omega^a - \frac{1}{l}\, e^a\right) T_a$.

Let us now look at the Euclidean continuation of the global de Sitter metric (7). This is obtained by a Wick rotation of the time coordinate $\tau_E = i\tau$. The period of τ_E is $2\pi\beta = 2\pi l$; obtained from the condition that the metric be regular everywhere. The metric is

$$ds^2 = l^2 d\tau_E^2 + l^2 \cos^2\tau_E (d\theta_1^2 + \sin^2\theta_1 d\theta_2^2) \tag{11}$$

This is clearly the metric on a three-sphere, and therefore the Euclidean de Sitter space topology is S^3.

One can also consider the Euclidean continuation of the metric on the static patch (8). This is obtained by taking $t_E = it$. The metric is

$$ds^2 \;=\; N^2\; dt_E^2 + N^{-2}\; dr^2 + r^2\; d\phi^2 \tag{12}$$

where N is given by (9). The period of the Euclidean time is $\beta = 2\pi l$, again obtained from the regularity of the metric. Since $0 \leq r \leq l$, we can make a change of coordinates $r = l\sin\theta$ and $\theta = \frac{\pi}{2}$ is the horizon. The metric is now

$$ds^2 \;=\; \cos^2\theta dt_E^2 + l^2 d\theta^2 + l^2\sin^2\theta d\phi^2 \tag{13}$$

which is again the metric on a three-sphere. The horizon is at $\theta = \frac{\pi}{2}$.

Thus, surprisingly, both the global de Sitter metric (7) and the static patch metric (8) seem to continue to the *same* Euclidean manifold. To see that the metric (12) is the same as the three-sphere metric, make a coordinate change from the static coordinates (t_E, r, ϕ) to the global coordinates $(\tau_E, \theta_1, \theta_2)$ through the transformations :

$$\begin{aligned} \phi &= \theta_2 \\ \frac{r}{l} &= \cos\tau_E \quad \sin\theta_1 \\ \tan\frac{t_E}{l} &= \frac{1}{\cos\theta_1} \quad \tan\tau_E \end{aligned} \tag{14}$$

These transformations cast the metric (12) into the more familiar form of the three-sphere metric (11). However, we recall that the ranges of the angles covering the entire three-sphere are $0 \leq \theta_2 \leq 2\pi$, and $0 \leq \tau_E, \theta_1 \leq \pi$. In (14), the static patch has the restriction $0 \leq r \leq l$ - this implies for τ_E a range $0 \leq \tau_E \leq \frac{\pi}{2}$ - half of the range required to cover the three-sphere. Thus the Euclidean continuation of the static patch metric is the metric on a three-sphere, but it does not cover the three-sphere completely. In fact, on Euclideanisation, the static patch has the topology of a solid torus. To see this, note that the topology described by the Lorentzian static patch is $D_2 \otimes R$, where R denotes the time direction t. On Euclideanisation, the time t becomes periodic, and therefore the topology should be $D_2 \otimes S^1$, a solid torus. The Euclidean continuation of each patch II and IV is a solid torus with metric (12). The Euclidean continuation must be understood as the gluing of the two solid tori in such a way that the resultant manifold is closed; and a three-sphere - the gluing is such that the metric now covers the three-sphere completely. This can be done easily by gluing the two solid tori with oppositely oriented boundaries after performing a modular transformation on the boundary of one of them.

We would like to compute the entropy of de Sitter space in a Euclidean formulation. Since the physical patch corresponding to a timelike observer is the static patch, we would like to work with the metric describing this patch, in which the horizon is manifest. However, we must consider only *one* of the patches II and IV. Thus we are interested in the Euclidean gravity partition function studied through two $SU(2)$ Chern-Simons theories on a solid torus. Corresponding to the metric (12), the connections for the two $SU(2)$ Chern-Simons theories are given by:

$$\begin{aligned} A^0 &= -N(d\phi + \frac{1}{l}dt_E) \\ A^1 &= \frac{1}{lN}dr \\ A^2 &= -\frac{r}{l^2}dt_E + \frac{r}{l}d\phi \end{aligned} \tag{15}$$

$$\begin{aligned} B^0 &= N(-d\phi + \frac{1}{l}dt_E) \\ B^1 &= -\frac{1}{lN}dr \\ B^2 &= -(\frac{r}{l^2}dt_E + \frac{r}{l}d\phi) \end{aligned} \tag{16}$$

4 Partition function for quantum gravity in de Sitter space

We use the Chern-Simons formulation of gravity to construct a partition function for gravity in de Sitter space. We saw in the last section that the manifold of interest is a solid torus. This is also the topology of the Euclidean BTZ black hole. This enables us to use the construction developed in [13] for Euclidean BTZ black holes here too.

In order to compute the de Sitter partition function, we first evaluate the Chern-Simons path integral on a solid torus. This path integral has been discussed in [18], [20], [21] and [22]. Through a suitable gauge transformation, the connection is set to a constant value on the toroidal boundary. In terms of coordinates on the toroidal boundary x and y with unit period, we can define $z = (x + \tau y)$ such that

$$\int_a dz = 1, \qquad \int_b dz = \tau \tag{17}$$

where a is the contractible cycle and b the non-contractible cycle of the solid torus and $\tau = \tau_1 + i\tau_2$ is the modular parameter of the boundary torus. Then, the connection can be written as [20]:

$$A = \left(\frac{-i\pi\tilde{u}}{\tau_2}\, d\bar{z} + \frac{i\pi u}{\tau_2}\, dz \right) T_3 \tag{18}$$

where u and $\tilde{u}$ are canonically conjugate fields and obey the canonical commutation relation:

$$[\tilde{u}, u] = \frac{2\tau_2}{\pi(k+2)} \tag{19}$$

Since A is an $SU(2)$ connection, $\tilde{u} = \bar{u}$, where $\bar{u}$ is the complex conjugate of u. For the case of the BTZ black hole, the information about the black hole parameters M and J is contained in the non-trivial holonomy of the connection A around the non-contractible cycle b. In the de Sitter case, the Euclidean time is associated with the noncontractible b cycle of the solid torus. To see why, we recall that the Lorentzian static patch has a topology $D_2 \otimes R$, where R corresponds to the world-line of the *timelike* observer at the centre of the disc. In the Euclidean continuation, this time direction R, on compactification is precisely the non-contractible cycle of the solid torus. The holonomy of the connection around the non-contractible cycle is therefore related to the period of the Euclidean time. Similarly, the holonomy around the contractible cycle is related to the periodicity of the ϕ coordinate. Traces of the holonomies around the contractible a cycle and non-contractible b cycle from connections in (15) are:

$$Tr(H_a) = 2\cos(\Theta), \qquad Tr(H_b) = 2\cos\left(\frac{2\pi\beta}{l}\right) \tag{20}$$

Classical de Sitter space corresponds to a value of $\Theta = 2\pi$ (i.e a spacetime with no defect). From (18) and (20), we have

$$A_z = \frac{-i\pi}{\tau_2}\, \tilde{u}, \qquad A_{\bar{z}} = \frac{i\pi}{\tau_2}\, u \tag{21}$$

with

$$u = \frac{1}{2\pi}\left(\Theta\tau - \frac{2\pi\beta}{l}\right), \qquad \tilde{u} = \frac{1}{2\pi}\left(\Theta\bar{\tau} - \frac{2\pi\beta}{l}\right) \tag{22}$$

Next we write the Chern-Simons path integral on a solid torus with a boundary modular parameter τ. For a fixed boundary value of the connection, i.e. a fixed value of u, this path integral is formally equivalent to a state $\psi_0(u,\tau)$ with no Wilson lines in the solid torus. The states corresponding to having closed Wilson lines (along the non-contractible cycle) carrying spin $j/2$ ($j \le k$) representations in the solid torus are given by [18], [20], [21], [22]:

$$\psi_j(u,\tau) = \exp\left\{ \frac{\pi k}{4\tau_2}\, u^2 \right\} \chi_j(u,\tau), \tag{23}$$

where χ_j are the Weyl-Kac characters for affine SU(2). which can be expressed in terms of the well-known Theta functions as

$$\chi_j(u,\tau) \;=\; \frac{\Theta^{(k+2)}_{j+1}(u,\tau,0) \;-\; \Theta^{(k+2)}_{-j-1}(u,\tau,0)}{\Theta^2_1(u,\tau,0) \;-\; \Theta^2_{-1}(u,\tau,0)} \tag{24}$$

where Theta functions are given by:

$$\Theta^k_\mu(u,\tau,z) \;=\; \exp(-2\pi ikz) \sum_{n\in\mathcal{Z}} \exp 2\pi ik\left[(n+\frac{\mu}{2k})^2\tau \;+\; (n+\frac{\mu}{2k})u\right] \tag{25}$$

As in the computation in [13] for the BTZ black hole, the de Sitter partition function is constructed from the boundary state $\psi_0(u,\tau)$. The construction is motivated by the following observations :

In the Chern-Simons functional integral over a solid torus, we shall integrate over all gauge connections with fixed holonomy H_b around the non-contractible cycle. This corresponds to the partition function with fixed period $2\pi\beta$ of the Euclidean time, that is, fixed inverse temperature. This in turn means we are dealing with the canonical ensemble. The variable conjugate to this holonomy is the holonomy around the other (contractible) cycle, which is *not* fixed any more to the classical value given by $\Theta = 2\pi$ for de Sitter space. We must sum over contributions from all possible values of Θ in the partition function. This corresponds to starting with the value of u for the classical solution, i.e. with $\Theta = 2\pi$ in (20), and then considering all other shifts of u of the form

$$u \;\rightarrow\; u + \alpha\tau \tag{26}$$

where α is an arbitrary number. This is implemented by a translation operator of the form

$$T \;=\; \exp\left(\alpha\tau\frac{\partial}{\partial u}\right) \tag{27}$$

However, this operator is not gauge invariant. The only gauge-invariant way of implementing these translations is through Verlinde operators of the form

$$W_j \;=\; \sum_{n\in\Lambda_j} \exp\left(\frac{-n\pi\bar{\tau}u}{\tau_2} + \frac{n\tau}{k+2}\frac{\partial}{\partial u}\right) \tag{28}$$

where $\Lambda_j \;=\; -j, -j+2, ..., j-2, j$. This means that all possible shifts in u are not allowed, and from considerations of gauge invariance, the only possible shifts are

$$u \;\rightarrow\; u + \frac{n\tau}{k+2} \tag{29}$$

where n is always an integer taking a maximum value of k. Thus, the only allowed values of Θ are $2\pi(1+\frac{n}{k+2})$. We know that acting on the state with no Wilson lines in the solid torus with the Verlinde operator W_j corresponds to inserting a Wilson line of spin $j/2$ around the non-contractible cycle. Thus, taking into account all states with different shifted values of u as in (29) means that we have to take into account all the states in the boundary corresponding to the insertion of such Wilson lines. These are the states $\psi_j(u,\tau)$ given in (23).

b) In order to obtain the final partition function, we must also integrate over all values of the modular parameter, i.e. over all inequivalent tori with the same holonomy around the non-contractible cycle. The integrand, which is a function of u and τ, must be the square of the partition function of a gauged $SU(2)_k$ Wess-Zumino model corresponding to the two $SU(2)$ Chern-Simons theories. It must be modular invariant – modular invariance corresponds to invariance under large diffeomorphisms of the torus. The partition function is then of the form

$$Z \;=\; \int d\mu(\tau,\bar{\tau}) \left| \sum_{j=0}^{k} \; a_j(\tau) \; \psi_j(u,\tau) \right|^2 \tag{30}$$

where $d\mu(\tau,\bar{\tau})$ is the modular invariant measure, and the integration is over a fundamental domain in the τ plane. Coefficients $a_j(\tau)$ must be chosen such that the integrand is modular invariant.

As discussed in [13], these coefficients are given by $a_j(\tau) \;=\; (\psi_j(0,\tau))^*$ so that the partition function is uniquely fixed to be

$$Z_{dS} = \int d\mu(\tau,\bar{\tau}) \left| \sum_{j=0}^{k} \; (\psi_j(0,\tau))^* \; \psi_j(u,\tau) \right|^2 \tag{31}$$

where the modular invariant measure is $d\mu(\tau,\bar{\tau}) \;=\; \frac{d\tau d\bar{\tau}}{\tau_2^2}$.

This is an *exact* expression for the canonical partition function of quantum gravity on dS_3. We now proceed to compute the partition function by substituting in the expression above the values of u and $\bar{u}$ from (22) with $\Theta = 2\pi$. We work in the regime where k (and therefore l) is large. Also, we must perform an analytic continuation to get the Lorentzian result - this is done by taking $G \to -G$, and $\beta \to i\beta$. For the regime when k is large, the leading contribution to the sum in the integrand comes from $j = 0$ as in [13]. The τ_2 integral can in fact be done exactly. We have

$$Z_{dS} = \int_{-1/2}^{1/2} d\tau_1 \; 4\pi \;\; e^{\beta \; k/2l} \;\; \frac{1}{f(\tau_1)} \;\; K_1(-k/2 \;\; f(\tau_1)) \tag{32}$$

where $f(\tau_1) = \sqrt{\frac{\beta^2}{l^2} - 4\pi^2\tau_1^2}$, and K_1 is the Bessel function of imaginary argument. Using the approximation for the Bessel function with large argument

$$K_1(z) = \sqrt{\frac{\pi}{2z}} e^{-z}[1 + O(\frac{1}{z}) + ...] \tag{33}$$

with the replacement $\beta = 2\pi l$ for de Sitter space, we get, in the large k regime :

$$Z_{dS} = 4\sqrt{\pi} \ \ \frac{4G}{2\pi l} \ \ e^{2\pi l/4G} \tag{34}$$

The form of the partition function indicates that at leading order, it is of the form e^S, where $S = \frac{2\pi l}{4G}$ is the semi-classical entropy. Since this is the partition function in the canonical ensemble, we would have expected an additional term $e^{-i\beta E}$ where E is the energy of de Sitter space. The notion of energy in asymptotically de Sitter spaces needs to be defined carefully, due to the absence of a global timelike Killing vector in de Sitter space. The energy E that emerges in our formalism is defined on the horizon, and not at asymptotic infinity, as has been done, for e.g in [5]. Our result seems to indicate that that energy E is zero for de Sitter space. Such a result coincides with the definition of energy as given by Abbott and Deser [23]. The canonical partition function at leading order is therefore the same as the density of states. The multiplicative prefactor in (34) is the leading correction to the semi-classical result. The entropy is therefore

$$S = (2\pi l)/4G \ - \ \log(2\pi l) + ... \tag{35}$$

The leading term is the semi-classical Bekenstein-Hawking entropy that is proportional to the horizon “area”. The second term is the leading correction that is logarithmic in area. In the following sections, we discuss in detail the results we have obtained - on the nature of the quantum fluctuations, and the logarithmic correction we have seen above.

5 The nature of the quantum fluctuations

In our set-up, from the choice of ensemble and considerations of gauge invariance, the partition function (31) involved insertion of closed Wilson lines of spins $j/2 \neq 0$, $j \leq k$. These correspond to defects centered at the origin of the ϕ coordinate in (12), i.e around the worldline of the timelike observer. As is well-known, such defects correspond to point masses in dS_3 [24]. A point mass in dS_3 can be described in static coordinates by the metric

$$ds^2 \ = \ - N^2 \ dt^2 + N^{-2} \ dr^2 + r^2 \ d\phi^2 \tag{36}$$

where now

$$N^2 \ = \ (8GM \ - \ \frac{r^2}{l^2}), \tag{37}$$

and $0 \leq r \leq r_+$, with $r_+ \ = \ l\sqrt{8GM}$ as the radius of the cosmological horizon. Let us consider the static patch metric (8) of dS_3. Near $r = 0$, the origin of the ϕ coordinate, it looks like

Minkowski space. Now, let us consider the metric (36) near $r = 0$. Then, it can be rewritten as the Minkowski metric

$$ds^2 = -dt_1^2 + dr_1^2 + r_1^2 d\phi_1^2 \tag{38}$$

where $t_1 = \sqrt{8GM}t$, $r_1 = r/\sqrt{8GM}$ and $\phi_1 = \sqrt{8GM}\phi$. But since ϕ had a periodicity 2π, ϕ_1 has a periodicity $2\pi\sqrt{8GM}$. Thus the deficit angle at $r = 0$ is $2\pi(1 - \sqrt{8GM})$, i.e $2\pi(1 - \frac{r_+}{l})$. When $r_+ = l$, the deficit is zero, and we have de Sitter space.

In the partition function (31), there is a discrete sum over deficits in the ϕ coordinate - from (29), the deficits are of the form $2\pi\ \frac{n}{k+2}$ where n is an integer taking the maximum value of k. $n = 0$ corresponds to de Sitter space, i.e no deficit. Thus from the discussion above, we see that $(1 - \frac{r_+}{l}) = \frac{n}{k+2}$. The maximum deficit possible is $n = k$. Thus there seems to be a maximum allowed value for the "mass" of the point particle, M. Interestingly, this relation also implies that r_+, and therefore the mass of the point particle takes only a *discrete* number of values, labelled by the integer n. This result is surprising, because classically, while the meaning of spacetimes with deficit angles greater than 2π is not clear, since the deficit angle is a continuous variable, all point masses with deficits ranging from 0 to 2π are allowed. However the gauge invariance and quantization prevents this.

There is another interesting observation : In the computation of the de Sitter partition function (31), we set $\Theta = 2\pi$ - which corresponds to de Sitter space. Thus the partition function describes quantum fluctuations on the de Sitter background. We could instead choose any value of Θ from 0 to 2π. Then the partition function would correspond to fluctuations around a background with the corresponding point mass. What does that partition function look like? The exercise done in the previous section can be repeated for this case - and surprisingly, the leading answer is the same! Quantum fluctuations of any such background are point masses taking the same set of discrete values as above. The leading order contribution to the partition function again comes from spin $j = 0$, which corresponds to empty de Sitter space. Thus, the quantum fluctuation around the point mass background which corresponds to *empty* de Sitter space dominates the partition function at the leading order. Doing the computation carefully (taking into account the changed value of the temperature), we find that to the leading order, the entropy is the same as that of de Sitter space, i.e $S = (2\pi l)/4G$. However, the logarithmic corrections to the entropy are more complicated now and carry the information about the point mass parameter. All this strongly suggests a quantization of point mass configurations in a quantum theory of gravity on dS_3. Also, since the leading contribution to the entropy always comes from empty de Sitter space, this presents an explicit realisation of the entropy bound of Bousso [25] in three dimensions.

6 The log(area) correction to the semi-classical entropy

We note that in the expression for entropy (35), the correction to the Bekenstein-Hawking entropy is logarithmic in area, with a coefficient -1. The logarithmic correction has been observed in many computations of black hole entropy in quantum gravity. It was first computed for the (3+1)-d Schwarzschild black hole in the quantum geometry formulation of gravity - where, for a large massive black hole, the next order log(area) correction had a numerical coefficient, $-3/2$ [17]. Subsequently, this correction (with the same numerical coefficient!) has been seen in computations of the (2+1)-d BTZ black hole entropy in various approaches [17], [13], leading to the question of whether this coefficient is universal. Below, we clarify several issues related to the universality of this coefficient. Incidentally, the logarithmic correction with a different coefficient was seen in a one-loop computation of the correction to the entropy of the BTZ black hole (of small horizon area) due to a scalar field [26]. However, our discussion of the logarithmic coefficient is the correction due to quantum gravity fluctuations, and distinct from corrections due to scalar fields or other matter coupled to the black hole.

The computation of BTZ black hole entropy in [13] was done in the same (Chern-Simons) formulation as the de Sitter case, and the numerical coefficient of the logarithmic term was $-3/2$, whereas for the de Sitter case, it is -1. This is somewhat puzzling at first glance. The black hole entropy was computed in the regime $r_+ >> l$, where r_+ is the black hole horizon radius and l is the AdS radius of curvature. Then, there was an integral over the modular parameter similar to (31). The saddle-point for τ_2, the imaginary part of the modular parameter occured when $\tau_2 = r_+/l$. Thus this was the regime when τ_2 was large. An interesting observation was made in [13] that replacing r_+/l in the black hole partition function by l/r_+, where now $r_+ << l$, the AdS gas partition function was obtained, with the coefficient of the correction being $+3/2$. This corresponds to a situation where the modular parameter $\tau_2 = r_+/l$ is small. What happens when $r_+ \sim l$, i.e $\tau_2 \sim 1$? In fact, this is very similar to the de Sitter case, since the de Sitter horizon radius is exactly l! The computation follows similar lines and leads to similar results. It can in fact be verified directly from (31) that the saddle-point is at $\tau_2 = 1$. Here, we see that the coefficient of the logarithmic correction is -1. Thus, the coefficient of the correction seems to depend on the regime one is looking at. When, as in the above case, there are two independent length parameters l and r_+, only for $r_+ >> l$ do we get the coefficient $-3/2$.

Summarising our result for BTZ blackhole we find:

$$\begin{array}{rcl} For\ \ r_+ >> l \quad S &=& \dfrac{2\pi r_+}{4G} - \dfrac{3}{2}\log\left(\dfrac{2\pi r_+}{4G}\right) + \cdots \\ r_+ = l \quad S &=& \dfrac{2\pi r_+}{4G} - \log\left(\dfrac{2\pi r_+}{4G}\right) + \cdots \\ r_+ << l \quad S &=& \dfrac{2\pi l^2}{4r_+G} + \dfrac{3}{2}\log\left(\dfrac{r_+}{l}\right) + \cdots \end{array} \tag{39}$$

where the last expression in (39) for $r_+ << \ l$ is the entropy of the AdS gas.

Let us examine (32) closer to understand how the coefficient of the logarithmic correction in the de Sitter case is -1. Using the asymptotic expansion of the Bessel function (33), we see that the τ_2 integration contributes a logarithmic term with a coefficient $-1/2$. The τ_1 integration also contributes the same and the coefficient is thus -1.

Entropy of de Sitter space can also studied from an alternative point of view by using dS/CFT correspondence [7]. In this framework all the information about quantum gravity in the bulk is expected to be contained in the conformal field theory at past or future infinity. This CFT has a central charge $c = 3l/2G$. As shown in [5], the eigenvalues of the Virasoro generators L_0 and $\bar{L}_0$ for de Sitter space are both equal to $l/8G$. Using the Rademacher expansion for modular forms, one can generalize the Cardy formula for growth of states in a CFT beyond the leading term. In [27], the sub-leading correction to the entropy of a BTZ black hole was determined from this generalisation. We use these results to find the sub-leading corrections to the de Sitter entropy from the dS/CFT correspondence. From [27], the entropy obtained from a CFT with a given the central charge c and eigenvalue of the Virasoro generator $L_0 \ = \ N$, is given by

$$S_1 = S_0 - 3/2 \log S_0 + \log c + ... \tag{40}$$

where $S_0 = 2\pi\sqrt{\frac{c}{6}(N - \frac{c}{24})}$. This is the contribution from the Virasoro generator L_0. There is a similar contribution S_2 associated with the Virasoro generator $\bar{L}_0$, given by replacing N in the above by $\bar{N}$, the eigenvalue of $\bar{L}_0$.

Substituting $c = 3l/2G$ and $N = \bar{N} = l/8G$ in the above, we see that

$$S = S_1 + S_2 = 2\pi l/4G - \log(2\pi l) + ... \tag{41}$$

with the same coefficient -1 for the logarithmic correction as that obtained from the gravity partition function (34) in (35). Here, the contribution from each of S_1 and S_2 to the logarithmic correction was $-1/2 \ \log(2\pi l)$.

Thus, the quantum gravity calculation of de Sitter entropy and the entropy computation from the asymptotic CFT agree even in the sub-leading correction to the Bekenstein-Hawking term.

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